# A criterion of $p$-nilpotency of finite groups with some weakly $s$-semipermutable subgroups

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**Abstract.** Let $H$ be a subgroup of a group $G$. $H$ is called a weakly $s$-semipermutable subgroup of $G$ if there are a subnormal subgroup $T$ of $G$ and an $s$-semipermutable subgroup $H_{ssG}$ in $G$ contained in $H$ such that $G = HT$ and $H \cap T \leq H_{ssG}$. In this paper, we got a criterion of $p$-nilpotency of $G$ by some weakly $s$-semipermutable subgroups.

**Keywords:** finite group, $p$-nilpotent group, weakly $s$-semipermutable subgroup.

## 1. Introduction

Throughout this paper, all groups are finite. Let $G$ be a group and $H$ a subgroup of $G$. Recall that $H$ is said to be $s$-permutable (or $s$-quasinormal, $\pi$-quasinormal) of $G$ if $H$ permutes with every Sylow subgroup of $G$; $H$ is called weakly $s$-permutable in $G$ if there is a subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq H_{sG}$, where $H_{sG}$ is the maximal $s$-permutable subgroup of $G$ contained in $H$; $H$ is said to be $s$-semipermutable in $G$ if $H$ permutes with every Sylow $p$-subgroup $G_p$ of $G$ with $(|H|, p) = 1$. More recently, the authors introduced in [4] the following concept.

**Definition 1.1.** *Let $H$ be a subgroup of a group $G$. $H$ is called a weakly $s$-semipermutable subgroup of $G$ if there are a subnormal subgroup $T$ of $G$ and an $s$-semipermutable subgroup $H_{ssG}$ in $G$ contained in $H$ such that $G = HT$ and $H \cap T \leq H_{ssG}$.*

Li et al. in [4] get the following result:

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**Theorem I.** Let $G$ be a group and let $P$ be a Sylow $p$-subgroup of $G$, where $p$ is the smallest prime dividing $|G|$. Suppose that $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ of order $|D|$ and every cyclic subgroup of $P$ of order 4 (if $P$ is non-abelian 2-subgroup and $|D| = 2$ ) is weakly $s$-semipermutable in $G$. Then $G$ is $p$-nilpotent.

A celebrated theorem of Frobenius [9, Satz. IV.5.8] asserts that $G$ is $p$-nilpotent if either $N_G(H)$ is $p$-nilpotent or $N_G(H)/C_G(H)$ is a $p$-subgroup for every non-identity $p$-subgroup $H$ of $G$. In this paper we replace some of the conditions of Frobenius' theorem and Theorem I, namely, $H$ is restricted to be a $p$-subgroup of a fixed order, the condition that $p$ is the smallest prime of $|G|$ is changed by either $N_G(H)$ is $p$-nilpotent or $N_G(H)/C_G(H)$ is a $p$-group and we assume that $H$ is a weakly $s$-semipermutable subgroup of $G$. Our main result is the following theorems, which can be considered as a complement to Frobenius' theorem and Theorem I with weakly $s$-semipermutable subgroups.

**Main result**

Let $p$ be an odd prime dividing the order of a group $G$ and $P$ a Sylow $p$-subgroup of $G$. If there is a subgroup $D$ of $P$ with $1 < |D| < |P|$ such that every subgroup $H$ of $P$ with order $|D|$ is weakly $s$-semipermutable in $G$ and either $N_G(H)$ is $p$-nilpotent or $N_G(H)/C_G(H)$ is a $p$-group, then $G$ is $p$-nilpotent.

**Remark 1.2.** Let $H$ be a $p$-subgroup of a group $G$. If $N_G(H)$ is $p$-nilpotent, then obviously $N_G(H)/C_G(H)$ is a $p$-group. But the converse is not true in general. For example, let $G = A_4 \times C_2$. Evidently $N_G(C_2)/C_G(C_2)$ is a 2-group, but $G = N_G(C_2)$ is not 2-nilpotent.

All unexplained notations and terminologies are standard. The reader is referred to [9] if necessary.

## 2. Preliminaries

In this section, we list some lemmas which will be useful for the proofs of our results.

**Lemma 2.1** ([4, Lemma 2.1])**.** *(a) An s-permutable subgroup of $G$ is subnormal in $G$.*

*(b) If $H \leq K \leq G$ and $H$ is s-permutable in $G$, then $H$ is s-permutable in $K$.*

*(c) Let $K \trianglelefteq G$. If $H$ is s-permutable in $G$, then $HK/K$ is s-permutable in $G/K$.*

*(d) If $P$ is an s-permutable p-subgroup of $G$ for some prime p, then $N_G(P) \geq O^p(G)$.*

**Lemma 2.2** ([4, Lemma 2.2])**.** *Let $G$ be a group. Suppose that $H$ is an s-semipermutable subgroup of $G$. Then*

*(1) If $H \leq K \leq G$, then $H$ is s-semipermutable in $K$.*

(2) *Let $N$ be a normal subgroup of $G$. If $H$ is a $p$-group for some prime $p \in \pi(G)$, then $HN/N$ is s-semipermutable in $G/N$.*

(3) *If $H \leq O_p(G)$, then $H$ is s-permutable in $G$.*

(4) *Suppose that $H$ is a $p$-subgroup of $G$ for some prime $p \in \pi(G)$ and $N$ is normal in $G$. Then $H \cap N$ is also an s-semipermutable subgroup of $G$.*

**Lemma 2.3** ([4, Lemma 2.3])**.** *Let $G$ be a group and $U$ a weakly s-semipermutable subgroup of $G$ and $N$ a normal subgroup of $G$. Then:*

(1) *If $U \leq H \leq G$, then $U$ is weakly s-semipermutable in $H$.*

(2) *Suppose that $U$ is a $p$-group for some prime $p$. If $N \leq U$, then $U/N$ is weakly s-semipermutable in $G/N$.*

(3) *Suppose that $U$ is a $p$-group for some prime $p$. and $N$ is a $p'$-subgroup. Then $UN/N$ is weakly s-semipermutable in $G/N$.*

(4) *If $U$ is a $p$-group for some prime $p$ and $N$ a normal subgroup of $G$ contained in $O^p(G)$, then $U \cap N$ is an s-semipermutable subgroup of $G$.*

**Lemma 2.4** ([9, III, 5.2 and IV, 5.4])**.** *Suppose $p$ is a prime and $G$ is not a $p$-nilpotent group but whose proper subgroups are all $p$-nilpotent. Then:*

($a$) *$G$ has a normal Sylow $p$-subgroup $P$ for some prime $p$ and $G = PQ$, where $Q$ is a non-normal cyclic $q$-subgroup for some prime $q \neq p$;*

($b$) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$;*

($c$) *The exponent of $P$ is $p$ or $4$.*

**Lemma 2.5.** *Let $G$ be a group and $P$ a $p$-subgroup of $G$, where $p$ is a prime divisor of $|G|$. Suppose that $N_G(P)/C_G(P)$ is a $p$-group. Let $N$ be a normal subgroup of $G$. If either $N$ is a $p'$-group or $N \leq P$, then $N_{G/N}(PN/N)/C_{G/N}(PN/N)$ is also a $p$-group.*

**Proof** Let $N_{G/N}(PN/N) = K/N$. Then $PN \trianglelefteq K$. By the hypothesis, $P$ is a Sylow $p$-subgroup of $NP$. Thus by the Frattini argument $K = N_K(P)PN = N_G(P)N \leq K$, that is, $N_{G/N}(PN/N) = N_G(P)N/N$. It follows from $N_G(P)/C_G(P)$ is a $p$-group that

$$N_G(P)N/N/C_G(P)N/N \simeq N_G(P)N/C_G(P)N \simeq N_G(P)/C_G(P)(N_G(P) \cap N)$$

is a $p$-group. Evidently $C_G(P)N/N \leq C_{G/N}(PN/N)$.

Hence $N_{G/N}(PN/N)/C_{G/N}(PN/N)$ is a $p$-group as desired.

**Lemma 2.6.** *Let $G$ be a group and $H$ a $p$-subgroup of $G$, where $p$ is a prime divisor of $|G|$. Let $M$ be a subgroup of $G$ containing $H$. If $N_G(H)/C_G(H)$ is a $p$-group, then $N_M(H)/C_M(H)$ is also a $p$-group.*

**Proof** Since $N_M(H)/C_M(H) = N_G(H) \cap M/C_G(H) \cap M \cong (N_G(H) \cap M)C_G(H)/C_G(H) \leq N_G(H)/C_G(H)$, the result is obvious.

## 3. Proof of main result

Let $H$ be a $p$-subgroup of $G$. By Remark 1.2., we only need to prove that if there is a subgroup $D$ of $P$ with $1 < |D| < |P|$ such that every subgroup $H$ of $P$ with order $|D|$ is weakly $s$-semipermutable in $G$ and $N_G(H)/C_G(H)$ is a $p$-group, then $G$ is $p$-nilpotent. Assume it is false and $G$ is a counterexample with minimal order. Now we derive a contradiction from the following several steps.

**Step 1.** $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Consider $G/O_{p'}(G)$. Then by Lemmas 2.3(3) and 2.5, we have $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. The choice of $G$ yields that $G/O_{p'}(G)$ is $p$-nilpotent, which implies that $G$ is $p$-nilpotent, a contradiction.

**Step 2.** Let $T$ be a subgroup of $G$ such that $P \leq T < G$, then $T$ is $p$-nilpotent.

Let $H$ be a subgroup of $T$ of order $|D|$. By hypothesis and Lemma 2.6, we have that $N_T(H)/C_T(H)$ is a $p$-group and by Lemma 2.3(1) $H$ is weakly $s$-semipermutable in $T$. Hence $T$ satisfies the hypothesis of the theorem. The minimality of $G$ forces that $T$ is $p$-nilpotent.

**Step 3.** If $|P : D| > p$, then every subgroup $H$ of $P$ of order $|D|$ is $s$-semipermutable in $G$ and $O^p(G) = G$.

Assume that $P$ has a subgroup $H$ such that $|H| = |D|$ and $H$ is not $s$-semipermutable in $G$. By hypothesis, there exists a subnormal subgroup $K$ of $G$ such that $G = HK$ and $H \cap K < H$. It follows that $K < G$. Let $M$ be a normal maximal subgroup of $G$ containing $K$. Then $|G : M| = p$. By Lemmas 2.3(1) and 2.6, the hypothesis is still true for $M$ since $|P : D| > p$. The minimal choice of $G$ implies that $M$ is $p$-nilpotent and so $G$ is $p$-nilpotent, a contradiction. Hence every subgroup $H$ of $P$ of order $|D|$ is $s$-semipermutable in $G$. Similarly, we have that $O^p(G) = G$.

**Step 4.** $|D| > p$.

Suppose that $|D| = p$. Since $G$ is not $p$-nilpotent, $G$ has, by [9, IV, 5.4], a $p$-closed Schmidt subgroup $K = [K_p]K_q$, where $K_p \leq P$. By hypothesis and Lemma 2.4, the exponent of $K_p$ is $p$. Now every subgroup $H$ of $K$ of order $p$ is weakly $s$-semipermutable in $K$ by Lemma 2.3 and $N_K(H)/C_K(H)$ is a $p$-group by Lemma 2.6. Let $x$ be an element in $K_p$ of order $p$. Then $\langle x\rangle$ is weakly $s$-semipermutable in $K$. Let $T$ be a subnormal subgroup of $K$ and $\langle x\rangle_{ssK}$ an $s$-semipermutable subgroup of $K$ contained in $\langle x\rangle$ such that $K = \langle x\rangle T$ and $\langle x\rangle \cap T \leq \langle x\rangle_{ssK}$. Since $O^p(K) = K$, $T = K$ and so $\langle x\rangle = \langle x\rangle_{ssK}$ is $s$-semipermutable in $K$. By Lemmas 2.2(3) and 2.1(d) $\langle x\rangle$ is normal in $K$. It follows that $N_K(\langle x\rangle)/C_K(\langle x\rangle) = K/C_K(\langle x\rangle)$ is a $p$-group. Hence $\langle x\rangle \in Z(K)$, which implies that every element of $K$ of order $p$ were in $Z(K)$, then $K$ would be $p$-nilpotent by [9, Satz IV.5.5], a contradiction. Hence $|D| > p$.

**Step 5.** $O_p(G) \neq 1$.

Since $G$ is not $p$-nilpotent, by the Glauberman-Thompson Theorem, $N_G(Z(J(P)))$ is not $p$-nilpotent, where $J(P)$ is the Thompson subgroup of $P$. Then $P \leq N_G(Z(J(P)))$. By Step 2, we have $N_G(Z(J(P))) = G$ and so $O_p(G) \neq 1$.

**Step 6.** Let $N$ be a minimal normal subgroup of $G$ contained in $O_p(G)$. Then $|N| < |D|$.

If $|N| = |D|$, then by hypothesis, $G/C_G(N) = N_G(N)/C_G(N)$ is a $p$-group. So $O^p(G) \leq C_G(N)$. If $N$ is a Sylow $p$-subgroup of $C_G(N)$, then by Schur-Zassenhaus Theorem, $N$ has a complement $K$ in $C_G(N)$ and so $C_G(N) = N \times K$. It follows that $K$ is normal in $G$, which contradicts Step 1. Hence $N$ is not a Sylow $p$-subgroup of $C_G(N)$. It is easy to see that $C_G(N)$ satisfies the hypothesis. If $C_G(N) < G$, by induction $C_G(N)$ is $p$-nilpotent and then $G$ is $p$-nilpotent, a contradiction. Hence $C_G(N) = G$, that is, $N \leq Z(G)$ and so $|N| = |D| = p$, which contradicts step 4.

Now suppose that $|N| > |D|$. Let $Q$ be a Sylow $q$-subgroup of $G$, where $q \neq p$ is a prime divisor of $|G|$. Consider $C_G(N)$ and $NQ$. Obviously $N \leq C_G(N)$, $N \leq NQ$ and $P \cap C_G(N)$ is a Sylow $p$-subgroup of $C_G(N)$. It is easy to see that $NQ$ and $C_G(N)$ satisfy the hypothesis of the theorem. If $C_G(N) = G$, then $|N| = p$ and so $|D| = 1$, contradicts the hypothesis. Hence $C_G(N) < G$ and then $C_G(N)$ is $p$-nilpotent by the choice of $G$. Hence Step 1 forces that $C_G(N) = N$. Now consider $NQ$. If $NQ < G$, then by induction $NQ$ is $p$-nilpotent and so $Q \leq C_G(N) = N$, a contradiction. Hence $NQ = G$. It follows that $N = P$ is a minimal normal subgroup of $G$. Let $H$ be a subgroup of $N$ with order $|D|$, $T$ a subnormal subgroup of $G$ and $H_{ssG}$ an $s$-semipermutable subgroup of $G$ contained in $H$ such that $G = HT$ and $H \cap T \leq H_{ssG}$. If $T < G$, then $G = NT$ and $N \cap T = 1$ for the minimality of $N$. Therefore $N = H$, contradicts $|N| > |D|$. Hence $H = H_{ssG}$ is $s$-semipermutable in $G$. By Lemma 2.2(3), $H$ is $s$-permutable in $G$ and $O^p(G) \leq N_G(H)$ follows from Lemma 2.1(d). Noticing that $H \trianglelefteq N$, $H$ is normal in $G$, contradicts the minimality of $N$ and the hypothesis $1 < |D|$. Hence $|N| < |D|$.

**Step 7.** $G$ has a maximal subgroup $M$ such that $G = MN$, $M \cap N = 1$. Furthermore, $C_G(N) = N = O_p(G)$ is the unique minimal normal subgroup of $G$.

By Step 6 and Lemma 2.3, it is easy to see that $G/N$ satisfies the hypothesis of the theorem, so the choice of $G$ yields that $G/N$ is $p$-nilpotent. The uniqueness of $N$ and $N \leq \Phi(G)$ are obvious. So $G$ has a maximal subgroup $M$ such that $G = MN$, $M \cap N = 1$ and $M \cong G/N$ is $p$-nilpotent. Since $O_p(G) \cap M$ is normalized by $N$ and $M$, the uniqueness of $N$ yields $O_p(G) \cap M = 1$ and so $N = O_p(G)$. Noticing that $G$ is $p$-solvable, $C_G(N) \leq N$ follows from Step 1 and [7, Theorem 9.3.1]. On the other hand, $N$ is abelian, so $C_G(N) = N$.

**Step 8.**$|N| = p$.

By Step 7, $P = N(P \cap M)$. If $|P : D| > p$, pick a maximal subgroup $N_1$ of $N$ and pick a subgroup $S_1$ of $M \cap P$ such that $H = N_1 S_1$ is of order $|H| = |D|$.

Then by Step 3 every subgroup $H$ of $P$ of order $|D|$ is $s$-semipermutable in $G$. So $N_1 = N \cap H$ is $s$-semipermutable in $G$ by Lemma 2.2(4). Furthermore, $N_1$ is $s$-permutable in $G$ by Lemma 2.2(3), Therefore, $N_1$ is normal in $G$ by Lemma 2.1(4) and Step 3. Then $N_1 = 1$ by the minimality of $N$ and so $|N| = p$.

Now assume that $|P : D| = p$. Since $P \cap M < P$, there exists a maximal subgroup $P_1$ of $P$ such that $P \cap M \leq P_1$. Hence $P_1 \cap N$ is $s$-semipermutable in $G$ by the unique minimality of $N$ and Lemma 2.3(4), then $P_1 \cap N$ is $s$-permutable in $G$ by Lemma 2.2(3). Hence $P_1 \cap N$ is normal in $G$ follows from Lemma 2.1 and the fact that $P_1 \cap N$ is normal in $P$. The minimality of $N$ implies that $P_1 \cap N = 1$. Hence $N$ has order $p$.

**Step 9.** The final contradiction.

By Step 8, $Aut(N)$ is a cyclic group of order $p-1$. For $G/N \cong N_G(N)/C_G(N) \lesssim Aut(N)$, we have that $P \leq C_G(N) = N$, contradicts Step 6 and the hypothesis. The final contradiction completes the proof. □

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